Parity-Time Symmetry Breaking beyond One Dimension: the Role of Degeneracy

Li Ge\textsuperscript{1,2} and A. Douglas Stone\textsuperscript{3}

\textsuperscript{1}Department of Engineering Science and Physics, College of Staten Island, CUNY, Staten Island, NY 10314, USA
\textsuperscript{2}The Graduate Center, CUNY, New York, NY 10016, USA
\textsuperscript{3}Department of Applied Physics, Yale University, New Haven, CT 06520-8482, USA
li.ge@csi.cuny.edu

Abstract—We consider the role of degeneracy in Parity-Time (PT) and Rotation-Time symmetry breaking for non-hermitian wave equations beyond one dimension. We show that the non-hermicity of the system eigenmodes can either onset abruptly as in typical PT-symmetric systems or be a linear function of the gain/loss parameter. These results are illustrated by using different T-breaking perturbations of a uniform dielectric disk and sphere, and a group theoretical analysis is given in the disk case.

Parity-Time (PT) symmetric systems have attracted considerable interest in the past few years. These are non-hermitian systems which are invariant under the combined action of a parity and time-reversal operation. In the case of closed Hamiltonian systems the transition is from a regime of real energy eigenvalues to complex conjugate pairs of eigenvalues as the degree of non-hermiticity is increased [1]. For the case of open, scattering systems, the transition is seen in the eigenvalues of the scattering matrix, which can remain on the unit circle despite the non-hermiticity up to some threshold and then depart from it in pairs with inverse moduli [2, 3]. In both cases the transition occurs when two eigenvalues coincide at an exceptional point (EP) which corresponds not to a degeneracy of the relevant operator but to a point at which it becomes defective (two eigenvectors coalesce), and hence is non-diagonalizable [4, 5, 6]. A major application of the theory of PT-symmetry breaking is to the wave equation of electromagnetism where the possibility of adding gain and loss in a PT-symmetric manner allows observation of many intriguing phenomena [7, 8, 9, 10, 11, 12].

Essentially all of the work on PT-symmetry breaking has focused on one-dimensional (1D) or quasi-1D (coupled waveguide) systems. These systems can never have a high enough symmetry group to generate generic degeneracies. In the current work we focus on two-dimensional (2D) and three-dimensional (3D) PT-symmetric scalar wave systems, described by the Helmholtz equation

\[ -\nabla^2 \phi(\vec{r}) = \left[ \epsilon_c(\vec{r}) + i\tau g(\vec{r}) \right] \frac{\omega^2}{c^2} \phi(\vec{r}), \]

which can have the new feature of continuous symmetries and generic degeneracies in the absence of the T-breaking non-hermitian perturbation. The cavity dielectric function \( \epsilon_c(\vec{r}) \), gain and loss strength \( \tau \), and their spatial profile \( g(\vec{r}) \) are real quantities. We adopt the convention that \( \tau \) is non-negative, with which \( g(\vec{r}) < 0 \) (\( > 0 \)) represents gain (loss).

As we show in Fig. 1(a), the PT-transition is absent in such systems if \( T \) is generically broken, meaning that they do not have a real spectrum even when the T-breaking is infinitesimal. However, if \( T \) is not generically broken, i.e. if some further discrete spatial symmetries are preserved, then it is possible that either the entire spectrum remains real over a finite interval (standard PT behavior; see Fig. 1(b)) or a finite subset of the degenerate spectrum does (see Fig. 1(c)).

These scenarios are analyzed using a coupled-mode theory and generalized “point groups” \( S \equiv \{ PT, \chi \} \), where \( \chi \) includes the identity operator 1. The examples examined in Fig. 1(b) and (c) have \( v \) angular blocks of equal area, and the corresponding \( S \) is a generalization of the dihedral group, \( D_{2v} \), describing the symmetries of regular polygons of \( v \) sides. \( D_{2v} \) contains \( 2v \) elements, including 1, \( v \) − 1 rotations, and \( v \) reflections. We denote the relevant generalization of this group to our system as \( D_{PT} \). The difference between \( D_{PT} \) and \( D_{2v} \) is due to the effect of \( T \) breaking while \( PT \) is preserved. One finds that \( v \) elements in \( D_{2v} \) are no longer symmetry operators, but become so again when multiplied by \( T \). This can be shown to be a general property of generalized point groups with \( PT \) symmetry. For the “PT-wheel” configuration shown in Fig. 1(c), \( S = \)
Figure 1: Different $PT$-transition scenarios in a 2D disk geometry. (a) Absence of $PT$ transitions with no additional discrete symmetry. (b) Protected $PT$ transitions and an entirely real spectrum at small $\tau$ with additional discrete symmetries. (c) Partial transitions with additional discrete symmetries. Insets show the corresponding gain (G) and loss (L) profiles.

$DT_8 \equiv \{1, PT, P_{\frac{\pi}{4}}, P_{\frac{\pi}{2}}T, R_{\phi}T, R_{\pi}, R_{\frac{3\pi}{2}}T\}$, and removing the 4 $T$-operators gives the original $D_8$ group. Here $R_{\phi}$ denotes clockwise rotation about the origin by $\phi$. The difference between this case that that in Fig. 1(b), described by $DT_{12}$, is the lack of an operator that decouples all pairs of degenerate modes in the disk geometry, which can be, for example, $P_{\frac{\pi}{2}}$ or $R_{\pi}T$. These operators protect $PT$-transitions at a finite $T$-breaking perturbation, without which some pairs of eigenmodes, if not all, acquire a finite non-hermicity at infinitesimal $\tau$ and be in the $PT$ broken phase. Interestingly, the dihedral group has another generalization with $PT$-symmetry, and the cyclic group can also be generalized.

In addition, our analysis shows that other composite symmetries which can occur in higher dimension, such as $RT$ where $R$ represents rotation by $\pi$, can behave differently from $PT$ and can exhibit a fully real spectrum when the corresponding $PT$ system does not. Our analysis also shows that it is possible for multimode coupling to restore the $PT$-symmetric phase, at finite $T$-breaking, if it is appropriately tuned. We thank Konstantinos Makris, Ramy El-Ganainy, Stefan Rotter, and Jan Wiersig for helpful discussions. This project was partially supported by PSC-CUNY 45 Research Grant and NSF under Grant No. ECCS 1068642.

REFERENCES