Metamaterials, a challenge for homogenization theory

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Metamaterials are crystal-like composites, derived by spatial translations from a generating cell, where high-frequency electromagnetic fields propagate in surprising ways at some frequencies: Negative refraction, "bending light the wrong way", refocusing of divergent beams into tightly parallel ones, are observed, and attributed to the fact that, at a spatial scale higher than the cell size, the composite behaves as a homogeneous material, but one with "emerging" properties, not possessed by any of the components.

More specifically, this material may have, when used in chunks much larger than its cell size, and within a specific frequency range, an "effective" permeability μ_{eff} and permittivity ε_{eff} (both complex and frequency-dependent) with *negative* real parts, hence an index $n = (\varepsilon\mu/\varepsilon_0\mu_0)^{1/2}$ which can have a negative real part, too. (Imaginary parts, which cause losses, and are negative, can be made relatively small, so there be little attenuation of propagating waves.) Hence the somewhat eery properties of these so-called "negative-index materials" (NIM). Such materials can be produced, and are actively studied: The prospect of, for instance, making quality lenses with only *plane* interfaces, is enough to justify this interest.

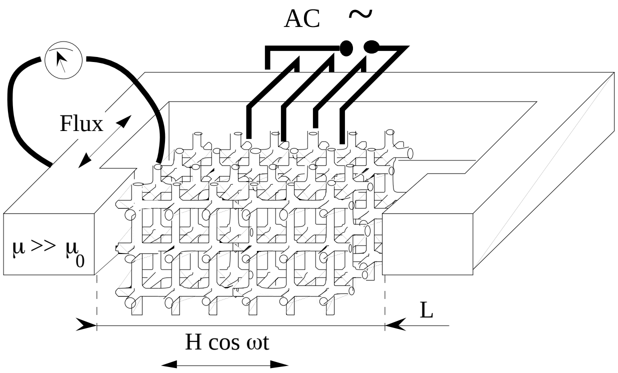

Fig. 1. Determining the effective permeability of a conductive grid, made of non-magnetic metal ($\mu = \mu_0$, $\varepsilon = \varepsilon_0 - i\sigma/\omega$). Average field H, at angular frequency ω, is forced through this material by Ampère-turns around the magnetic circuit, in which an induction $\mathrm{Re}[\mu_{eff} H \exp(i\omega t)]$ can be measured (as sketched in the left of the picture). The effective permeability μ_{eff} thus obtained is complex, with real part *lower* than μ_0: A "permittivity contrast" (between ε and the ε_0 of the air) thus induces diamagnetism—a well-known phenomenon, that standard, "static" homogenization (see text) cannot account for.

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For a very simple example, consider the situation of Fig. 1: An AC field H is forced through a sample of metamaterial, and the resulting flux in the laminated iron core is measured. With proper correction for edge effects (which post-homogenization numerics will provide), this setup allows one to determine μ_{eff}. The real part of this "homogenized" permeability is found to be lower than μ_0, which is not surprising: Within the right frequency window, skin effect excludes the magnetic field from the conductive part, hence a smaller induction flux, for a given H, than in air.

One might feel like predicting this effect by using classical homogenization theory [1], along the following lines. Let us cast the problem as

(1) $$-i\omega\,\varepsilon\, e + \mathrm{rot}\, h = j^s, \quad i\omega\,\mu h + \mathrm{rot}\, e = 0,$$

in all space, with source current j^s. To account for the fact that coefficients $\varepsilon \equiv \varepsilon_0 - i\sigma/\omega$ and (possibly) μ vary rapidly with position x in the region M occupied by the metamaterial, we assume they are "C_α-periodic" inside M, meaning that $\varepsilon(x + \alpha v_i) = \varepsilon(x)$ for x inside M and all three vectors v_i that subtend the "master cell" (Fig. 2) from which the metamaterial is generated. The solution $\{e_\alpha, h_\alpha\}$ of (1) then depends on the small non-dimensional parameter α, and the purpose of the theory is to show the existence, when $\alpha \to 0$, of an appropriately weak limit $\{e_0, h_0\}$, to be characterized as the solution of some partial differential equations system.

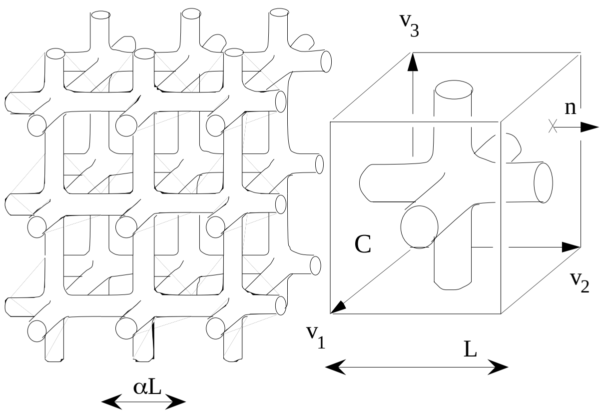

Fig. 2. Left: part of the grid, as a union of congruent homothetic images of the "master cell", right. The latter is taken of size L, comparable to the device's dimensions in Fig. 1. The actual size of a generic cell of M is then αL, with small α.

As proved under more general assumptions in [2], this "limit problem" is

(2) $\quad -i\omega\,\varepsilon_{eff}\, e_0 + \text{rot}\, h_0 = j^s, \quad i\omega\,\mu_{eff}\, h_0 + \text{rot}\, e_0 = 0,$

where the linear operators ε_{eff} and μ_{eff} are obtained by solving the "cell problem"

(3) $\quad \overline{\varepsilon}_{eff}\, \overline{E} \cdot E = \text{argstat}\{\int_{C_\alpha} \overline{\varepsilon}\, |E + \text{grad}\, \psi|^2\}/\text{vol}(C_\alpha)$

and its homologue in H,

(4) $\quad \overline{\mu}_{eff}\, \overline{H} \cdot H = \text{argstat}\{\int_{C_\alpha} \overline{\mu}\, |H + \text{grad}\, \varphi|^2\}/\text{vol}(C_\alpha).$

Let's decipher this: All quantities are complex valued, with an overbar for conjugation; E and H are ordinary 3D vectors ($\cdot$ denotes the dot product), ψ and φ are C_α-periodic potentials, and "argstat" means "stationarize the right-hand side with respect to the potential and get the corresponding critical value"; this value is then a quadratic form of the vector parameter E or H, the 3×3 matrix of which is, up to conjugation, the sought-for ε_{eff} or μ_{eff}.

One can check, by treating examples where ε or μ oscillate in one spatial direction only, that (3) and (4) consist in taking *averages*, arithmetic of harmonic as the case may be, of ε and μ in C. So the procedure is a kind of sophisticated, "energy-oriented", averaging of ε and μ. (Which is already worrisome: What hope is there to get the *negative* values characteristic of NIMs by averaging essentially *positive* coefficients?) Clearly, anisotropy can occur (this depends on the symmetries, or lack thereof, of the cell), and complex-valued ε and μ result in complex-valued effectives. But it's also plain that (3) *and* (4) *are decoupled* problems: Therefore, periodic variations of σ, i.e., of ε, *cannot affect* μ_{eff}, contrary to the evidence offered by the Fig. 1 setup. Note also that ε_{eff} and μ_{eff} do not depend on ω, again, contrary to observations.

Yet, this is backed by a convergence *theorem*, so what went wrong? We have a family of problems indexed by a parameter α, there is a limit when $\alpha \to 0$, and since the actual, real-life value of α we have at hand is *small*, we should be justified in taking $\{e_0, h_0\}$, for all purposes, as an approximation to $\{e_0, h_0\}$. In particular, the μ_{eff} provided by the cell-problem (4), which one can denote by μ^0 (a scalar in the isotropic case of Fig. 2) can serve as estimate of the μ^α that an accurate numerical simulation of the experiment of Fig. 1, should we have the power and means to tackle it, would give. Or can it?

That this reasoning if off the mark can be seen by thinking about the role played by skin effect in the experiment of Fig. 1. Flux is excluded from the metal because the ratio $\delta/\alpha L$ of skin depth to cell size is *small*, for actual values of the parameters. But by embedding the problem in the α-indexed family, we lose this essential qualitative feature when $\alpha \to 0$, whereas it should be preserved in order to capture the induced diamagnetism phenomenon.

How can that be done? Remark that skin effect results from the "contrast" between the values σ and $i\omega\,\varepsilon_0$ of "conductivity" of the metal and the air around, with a small value of $\beta = (\omega\,\varepsilon_0/\sigma)^2$, where the power 2 is for future convenience. So there are, actually, *two* small parameters, α and β, to consider in the modelling, and the actual field, $\{e_{\alpha\beta}, h_{\alpha\beta}\}$, is one in a doubly-indexed family. Looking for the limit when both tend to 0, we realize that there is no such thing: $\{0, 0\}$ is a singular point of the map $\{\alpha, \beta\} \to \mu^{\alpha\beta}$, which behaves as suggested by Fig. 3. Since the feature to be modelled is linked with the ratio δ/α, the proper one-parameter family in which to embed the actual problem is characterized by a constant value (the physically realized one) of the ratio β/α.

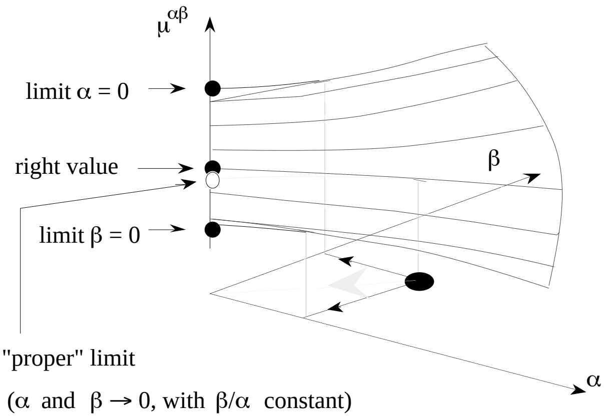

Fig. 3. Typical behavior of observed μ as a function of cell size (α) and contrast (β). The non-continuity at $\{0, 0\}$ defeats classical homogenization.

So again, we may investigate the limit, and establish a convergence theorem. But the cell-problem this points to is very different from (3)(4). It reads

(5) $-i\omega\,\varepsilon\,(e - i\omega/2\; B \times x) + \text{rot}(h + i\omega/2\; D \times x) = 0,$

(6) $\; i\omega\,\mu\,(h + i\omega/2\; D \times x) + \text{rot}(e - i\omega/2\; B \times x) = 0,$

where $B \times x$ is short for the vector field $x \to B \times x$, with B and D two 3D-vector parameters, and e and h C_α-periodic. Taking the mean value of e and h over the cell, one gets E and H, which relates to D and B via a 6×6 matrix. Hence, as will be developed, the expected "emergent properties" of meta-materials: cross-dependence of B and D on E and H respectively ("chirality") and possibility of negative effective partameters. The latter comes from the fact that the Lagrangian underlying (5)(6) is the *difference* between magnetic and electric energy, hence a non-definite quadratic form.

References

[1] A. Bensoussan, J.L. Lions, G. Papanicolaou: **Asymptotic methods for periodic structures**, North Holland (Amsterdam), 1978.

[2] A. Bossavit, G. Griso, B. Miara: "Modelling of periodic electromagnetic structures: bianisotropic materials with memory effects", **J. Math. Pures & Appl., 84** (2005), pp. 819-50.